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Improved Direct Solution of the Global Positioning System Equation

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A known direct solution of the global positioning system (GPS) equation, which yields user position and clock bias, is examined and tested using real GPS data. A comparison of the position computed using this solution with that obtained using the popular iterated least-squares solution reveals that the direct solution is in error by up to 4 m. An improved direct solution is presented, which takes into account the noise associated with the pseudorange measurements. The position error obtained when using this solution is an order of magnitude smaller than that generated by the direct solution. Finally, an iterated solution, which is based on the improved direct solution, is presented. The accuracy of the latter is comparable to that of the iterated least-squares solution of the GPS equation. The advantage of the new algorithm over that of the iterated least-squares solution is in its fast convergence rate. This advantage could be particularly important in space missions, where the initial guess may be far off from the real position.

I. Introduction

THE NAVSTAR global positioning system (GPS) is a satellite-based radio-navigation system that provides accurate three-dimensional position and precise time continuously and globally. To solve the three-dimensional position and the user clock bias, whose estimate yields precise time, we need at least four satellites.

The equations needed to be solved to find a user's position and clock bias from GPS range measurements are

$$(s_{i1} - x)^2 + (s_{i2} - y)^2 + (s_{i3} - z)^2 = (r_i - \tau)^2 \qquad i = 1, 2, 3, \ldots, n \tag{1}$$

where

n = number of observed GPS satellites
x, y, z = unknown user coordinates
s_{i1}, s_{i2}, s_{i3} = ith GPS satellite coordinates
τ = unknown receiver clock bias converted to distance
r_i = range to the ith GPS satellite

Because the number of unknowns (x, y, z, τ) is four, four satellites that do not lie in one plane are sufficient to solve Eqs. (1) for position. Usually more than four satellites are received at any particular time, and GPS receivers use them all to obtain a least-squares fit of the four unknowns. Because Eqs. (1) are nonlinear, the standard solution is an iterated least squares (ILS) solution. This algorithm is well known (see, e.g., Ref. 1). For the sake of comparison with the new algorithms that will be presented in the ensuing, we will now describe this solution.

Equations (1) are true for the ideal case. In fact, in such a case any four satellites will provide the same solution. In reality, however, the range measurements contain errors and the problem turns into a stochastic one. In such a case, Eqs. (1) take the form

$$\boldsymbol{R} = \boldsymbol{f}(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \boldsymbol{X}) + \boldsymbol{v} \tag{2}$$

where

$\boldsymbol{R}$ = vector of measured ranges to all satellites $[r_1, r_2, \ldots, r_n]^T$
$\boldsymbol{f}(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \boldsymbol{X})$ = $[(s_{i1} - x)^2 + (s_{i2} - y)^2 + (s_{i3} - z)^2]^{1/2} + \tau$
$\boldsymbol{S}_i$ = ith satellite position vector, $[s_{i1}, s_{i2}, s_{i3}]^T$
$\boldsymbol{X}$ = vector of the unknowns (three position components plus clock bias), $[x, y, z, \tau]^T$
$\boldsymbol{v} \in R^n$ = vector of measurement noises

Linearization of Eq. (2) about the nominal values $\boldsymbol{R}^*$ and $\boldsymbol{X}^*$ yields

$$\delta\boldsymbol{R} = F(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \boldsymbol{X}^*)\delta\boldsymbol{X} + \boldsymbol{v} \tag{3}$$

where

$$\delta\boldsymbol{R} = \boldsymbol{R} - \boldsymbol{R}^* = \boldsymbol{R} - \boldsymbol{f}(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \boldsymbol{X}^*) \tag{4}$$

$$\delta\boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}^* \tag{5}$$

and

$$F(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \boldsymbol{X}^*) = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \boldsymbol{X})}{\partial \boldsymbol{X}}\right|_{X = X^*} \tag{6}$$

This linearization leads to the following commonly used ILS algorithm.

1) Use an initial estimate, $\boldsymbol{X}_0^*$, of $\boldsymbol{X}$ to compute F according to Eq. (6).

2) Compute a weight matrix W based on the satellite signal-to-noise ratio.

3) Use $\boldsymbol{R}$ (measurements) and $\boldsymbol{X}_0^*$ to compute $\delta\boldsymbol{R}$ according to Eq. (4).

4) Compute a weighted-least-squares estimate of $\delta\boldsymbol{X}$ as follows[2]:

$$\delta\boldsymbol{X} = (F^T W^{-1} F)^{-1} F^T W^{-1} \delta\boldsymbol{R} \tag{7}$$

5) Update $\boldsymbol{X}_0^*$:

$$\boldsymbol{X}_1^* = \boldsymbol{X}_0^* + \delta\boldsymbol{X} \tag{8}$$

6) Check the convergence test:

$$|\delta\boldsymbol{X}| < \varepsilon \tag{9}$$

7) If the condition of Eq. (9) is met, stop. Otherwise, return to step 1 and increase the iteration index by 1.

Note that the Earth is turning during the time that elapses between the moment the satellites transmit their signals and the time each signal arrives at the receiver (pseudorange). This turn has to be compensated for.

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The value of ε and the initial guess of position determine the number of iterations needed for convergence. For a bad guess and a fast-moving vehicle such as spacecraft, the convergence time may be prohibitive. This motivates the examination of closed-form solutions of the GPS equation given in Eqs. (1).

Several direct solutions to the GPS navigation equations were introduced in the past,[3–6] each having different advantages; however, all of those solutions were based on the assumption that the GPS signals were deterministic. In other words, the measurements were assumed to be noiseless. In the present work we present direct solutions that treat the realistic case of noisy measurements.

In the next section we discuss a certain direct solution and results of a test we ran with this algorithm. Then, in Sec. III we introduce an algorithm that is an improvement on this solution, and an enhancement to the improved algorithm is presented in Sec. IV. A summary of the work reported on is given in the last section.

II. Direct Solution

A. Solution

Let us now examine a particular direct solution.[3] It can be easily shown that Eqs. (1) can be written as follows:

$$-2s_{i1}x - 2s_{i2}y - 2s_{i3}z + 2r_i\tau = r_i^2 - r_{si}^2 - \chi, \qquad i = 1, 2, \ldots, n \tag{10}$$

where

$$r_{si}^2 = \sum_{j=1}^{3} s_{ij}^2 \tag{11a}$$

$$\chi = x^2 + y^2 + z^2 - \tau^2 \tag{11b}$$

Now Eqs. (10) can be written as

$$H\boldsymbol{X} = \boldsymbol{R}_a + \chi\boldsymbol{R}_b \tag{12}$$

where

$$H = \begin{bmatrix} -2s_{11} - 2s_{12} - 2s_{13} 2r_1 \\ -2s_{21} - 2s_{22} - 2s_{23} 2r_2 \\ \vdots \quad \vdots \quad \vdots \\ -2s_{n1} - 2s_{n2} - 2s_{n3} 2r_n \end{bmatrix} \tag{13a}$$

$$\boldsymbol{R}_a = \begin{bmatrix} r_1^2 - r_{s1}^2 \\ r_2^2 - r_{s2}^2 \\ \cdot \\ \cdot \\ r_n^2 - r_{sn}^2 \end{bmatrix} \tag{13b}$$

$$\boldsymbol{R}_b = \begin{bmatrix} -1 \\ -1 \\ \cdot \\ \cdot \\ -1 \end{bmatrix} \tag{13c}$$

and

$$\boldsymbol{X}^T = [x, y, z, \tau] \tag{13d}$$

Define the pseudoinverse matrix $H^\dagger$

$$H^\dagger = (H^T H)^{-1} H^T \tag{14}$$

and left-multiply Eq. (12) by $H^\dagger$ to obtain

$$\boldsymbol{X} = H^\dagger \boldsymbol{R}_a + \chi H^\dagger \boldsymbol{R}_b \tag{15}$$

Note that, because $\chi = x^2 + y^2 + z^2 - \tau^2$ (that is, because χ is a scalar function of the unknowns), $\boldsymbol{X}$ cannot yet be determined from Eq. (15). Therefore, next we compute χ. First, for simplicity of notation, define the known vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ as follows:

$$\boldsymbol{p} = H^\dagger \boldsymbol{R}_a \tag{16a}$$

$$\boldsymbol{q} = H^\dagger \boldsymbol{R}_b \tag{16b}$$

Then Eq. (15) can be written as

$$\boldsymbol{X} = \boldsymbol{p} + \chi \boldsymbol{q} \tag{17}$$

(Note that $\boldsymbol{p} \in R^4$ and also $\boldsymbol{q} \in R^4$.) Second, use the components of $\boldsymbol{X}$ given in Eq. (17) to evaluate χ given in Eq. (11b). This yields

$$(p_1 + \chi q_1)^2 + (p_2 + \chi q_2)^2 + (p_3 + \chi q_3)^2 - (p_4 + \chi q_4)^2 = \chi \tag{18}$$

where p_i and q_i, $i = 1, 2, 3, 4$, are the components of $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively.

Equation (18) is a quadratic equation in χ. It can be written as

$$a\chi^2 + b\chi + c = 0 \tag{19}$$

where

$$a = q_1^2 + q_2^2 + q_3^2 - q_4^2 \tag{20a}$$

$$b = 2p_1q_1 + 2p_2q_2 + 2p_3q_3 - 2p_4q_4 - 1 \tag{20b}$$

$$c = p_1^2 + p_2^2 + p_3^2 - p_4^2 \tag{20c}$$

Finally, Eq. (19) is solved, which yields two solutions, one of which is not feasible. The solutions are then substituted into Eq. (17), which yields $\boldsymbol{X}_{1,2}$, and the latter are substituted into Eqs. (10). Only the feasible solution will satisfy the latter equation. [Note that the solution of Eq. (19) requires the knowledge of the coefficients a, b, and c, which are not functions of the unknown χ or $\boldsymbol{X}$; thus, we do not need any guess of $\boldsymbol{X}$ and the solution is, indeed, a direct one.]

In summary, to obtain the direct solution, the following has to be performed.

1) Compute r_{si}^2, the distance of each satellite from the origin.
2) Form H, $\boldsymbol{R}_a$, and $\boldsymbol{R}_b$.
3) Compute the pseudoinverse matrix $H^\dagger$.
4) Compute $\boldsymbol{p}$ and $\boldsymbol{q}$.
5) Compute the coefficients a, b, and c of the quadratic equation.
6) Compute $\chi_{1,2}$ by solving Eq. (19).
7) Substitute $\chi_{1,2}$ into Eq. (17) to obtain $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.
8) Substitute $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ into Eqs. (10). Only the feasible solution will satisfy the equation.

Note that, theoretically, there may be a case where there is no solution to the GPS equations. This will happen when the solution of Eq. (19) is not real. In practice, however, for terrestrial navigation and with reasonable range measurement errors, this will not happen.

B. Discussion

The direct solution was applied to raw data received by a real GPS receiver (Trimble 4000RL II). The antenna was placed at a point whose coordinates were latitude 32° 46′ 28.49370″ (N) and longitude 35° 01′ 20.58762″ (E), and $H = +255.7$ m mean sea level. The data were recorded on June 12, 1996, starting at 00:00 h universal time constant (UTC). The satellites involved in the solution were PRN 18, 27, 2, 19, 16, and 31. The position dilution of precision at 00:00 h was 2.5, and at 01:00 h it was 2.6. The selective availability (SA) disturbance was active. The result obtained using the direct solution was compared to the ILS solution, which served as a reference. (The reason for the latter is that the ILS solution is an accurate solution even though it is computed iteratively. One may be tempted to use a surveyed position as the reference, but even the most accurate algorithm for solving the navigation equations will differ from the true positions due to errors that are entirely independent of the algorithm, such as SA, ionospheric disturbance, etc.) The difference in the position computation between the direct solution and the ILS solution is shown in Fig. 1. As can be seen, the average difference was more than 2 m, and the difference itself was larger than 4.5 m at the end of the run. The difference stems from the fact that the ILS solution hinges on the pseudoinverse solution presented in Eq. (7) that yields the minimum-least-squares solution of $\delta\boldsymbol{X}$, which is the unknown vector in the linearized equation (7). In contrast, the direct solution hinges on the pseudoinverse solution computed in Eq. (14) and used in Eq. (15). This pseudoinverse is not known to be the optimal solution. More specifically, for Eqs. (1) to truly represent the relations between the measurements and the unknowns,

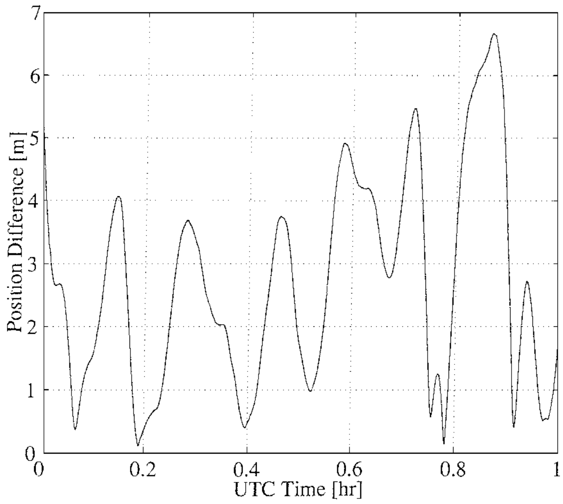

Fig. 1 Position difference between the direct and the ILS solutions: June 12, 1996.

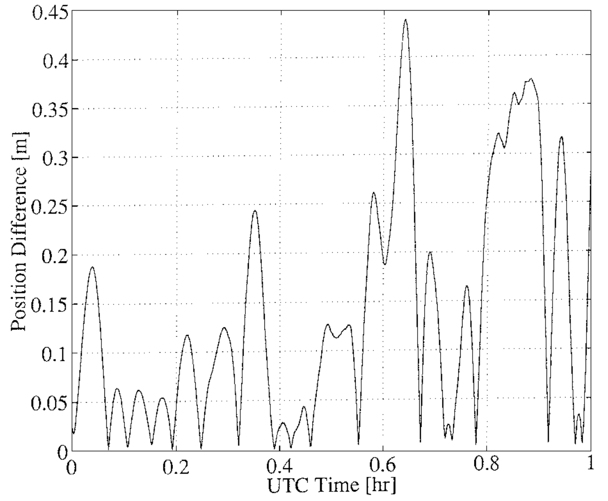

Fig. 2 Position difference between the IDS and the ILS solutions: June 12, 1996.

one has to include the additive error involved in the pseudorange measurements. When this is done, Eqs. (1) become

$$(s_{i1} - x)^2 + (s_{i2} - y)^2 + (s_{i3} - z)^2 = (r_i + v_i - \tau)^2 \quad (21)$$

which, like Eqs. (10), can be written as

$$-2s_{i1}x - 2s_{i2}y - 2s_{i3}z + 2(r_i + v_i)\tau = (r_i + v_i)^2 - r_{si}^2 - \chi \quad (22)$$

where r_{si}^2 and χ are as defined in Eqs. (11). Further manipulation of Eqs. (22) yields

$$-2s_{i1}x - 2s_{i2}y - 2s_{i3}z + 2r_i\tau = r_i^2 - r_{si}^2 - \chi + 2r_iv_i - 2\tau v_i + v_i^2 \quad (23)$$

It is realized that Eqs. (23) are not of the form of Eq. (3) for which Eq. (7) was the appropriate solution. Therefore, the use of the pseudoinverse of Eq. (14) for solving Eqs. (10) is not justified in the light of Eq. (23). This point will be discussed further in the next section.

III. Improved Direct Solution

Consider the order of magnitude of the last three terms of Eqs. (23). It is realized that r_i is of the order 10^7 m (pseudorange), τ is of the order 10^5 m (clock bias), and v_i is of the order 10^2 (SA). Consequently, $2r_iv_i$ is of the order 10^9 m^2, $2\tau v_i$ is of the order 10^7 m^2, and v_i^2 is of the order 10^4 m^2. Therefore, we choose to neglect in Eqs. (23) the terms $2\tau v_i$ and v_i^2, which yields

$$-2s_{i1}x - 2s_{i2}y - 2s_{i3}z + 2r_i\tau = r_i^2 - r_{si}^2 - \chi + 2r_iv_i \quad (24)$$

Equations (24) can be written as [see the passage from Eqs. (10) to Eq. (12)]

$$HX = R_a + \chi R_b + Gv \quad (25)$$

where H, R_a, and R_b are given in Eq. (13) and G is the diagonal matrix

$$G = \text{diag}\{2r_1, 2r_2, \ldots, 2r_n\} \quad (26)$$

Because G has an inverse, we can write Eq. (25) as

$$G^{-1}(R_a + \chi R_b) = G^{-1}HX - v \quad (27)$$

A least-squares estimate of X is the $\hat{X}$ that minimizes the cost function

$$J = \left[G^{-1}(R_a + \chi R_b) - G^{-1}HX\right]^T \left[G^{-1}(R_a + \chi R_b) - G^{-1}HX\right] \quad (28)$$

It can be easily shown that[2]

$$\hat{X} = [H^T(G^{-1})^T G^{-1} H]^{-1} H^T (G^{-1})^T G^{-1}(R_a + \chi R_b) \quad (29)$$

Because G is diagonal, Eq. (29) can be written as

$$\hat{X} = [H^T G^{-2} H]^{-1} H^T G^{-2}(R_a + \chi R_b) \quad (30)$$

Let p and q now be

$$p = [H^T G^{-2} H]^{-1} H^T G^{-2} R_a \quad (31a)$$

$$q = [H^T G^{-2} H]^{-1} H^T G^{-2} R_b \quad (31b)$$

then Eq. (30) can be written as

$$\hat{X} = p + \chi q \quad (32)$$

which is identical in form to Eq. (17). From this point on, the computation proceeds along the algorithm of the direct solution described earlier. We can now fully appreciate the difference between the direct solution and the preceding algorithm, which we name the improved direct solution (IDS). A comparison between Eqs. (12) and (14) indicates that, in the direct solution, the pseudoinverse was used merely to extract X from Eq. (12). It was not related to any error, nor was it designed to obtain an optimal solution, whereas in the IDS, the inverse was computed to obtain a least-squares solution. Indeed, as can be seen in Fig. 2, when the IDS was applied to the same data used before to examine the direct solution, the results were an order of magnitude better.

IV. IDS-Based Iterative Solution

An algorithm that is even more accurate than the IDS is an iterative algorithm that is based on the IDS. Its accuracy is comparable to that of the ILS, but its convergence is much faster. We start the exposition of the IDS-based iterative solution (IDSBIS) by expressing X in the form

$$X = \hat{X} + \Delta X \quad (33)$$

where $\hat{X}$ is the estimate of X and ΔX is the estimation error. When the components of Eq. (33) are substituted into Eqs. (21), the following is obtained:

$$(s_{i1} - \hat{x} - \Delta x)^2 + (s_{i2} - \hat{y} - \Delta y)^2 + (s_{i3} - \hat{z} - \Delta z)^2 = (r_i + v_i - \hat{\tau} - \Delta\tau)^2 \quad (34)$$

Equations (34) can be written as

$$-2(s_{i1} - \hat{x})\Delta x - 2(s_{i2} - \hat{y})\Delta y - 2(s_{i3} - \hat{z})\Delta z + 2(r_i - \hat{\tau})\Delta\tau = (r_i - \hat{\tau})^2 - (s_{i1} - \hat{x})^2 - (s_{i2} - \hat{y})^2 - (s_{i3} - \hat{z})^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta\tau^2) + 2(r_i - \hat{\tau})v_i - 2\Delta\tau v_i + v_i^2 \quad (35)$$

Define

$$\tilde{s}_{i1} = s_{i1} - \hat{x} \tag{36a}$$

$$\tilde{s}_{i2} = s_{i2} - \hat{y} \tag{36b}$$

$$\tilde{s}_{i3} = s_{i3} - \hat{z} \tag{36c}$$

$$\tilde{r}_{si}^2 = \sum_{j=1}^{3} \tilde{s}_{ij}^2 \tag{36d}$$

$$\tilde{r}_i = r_i - \hat{\tau} \tag{37a}$$

$$\Delta\chi = \Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta\tau^2 \tag{37b}$$

then Eqs. (35) can be written as

$$-2\tilde{s}_{i1}\Delta x - 2\tilde{s}_{i2}\Delta y - 2\tilde{s}_{i3}\Delta z + 2\tilde{r}_i\Delta\tau$$
$$= \tilde{r}_i^2 - \tilde{r}_{si}^2 - \Delta\chi + 2\tilde{r}_i v_i - 2\Delta\tau v_i + v_i^2 \tag{38}$$

The similarity of Eqs. (38) to Eqs. (23) is obvious; therefore, similar to the development following Eqs. (23), define $\tilde{H}$ as H in Eq. (13a), where s_{ij} is replaced by $\tilde{s}_{ij}$ and r_i is replaced by $\tilde{r}_i$. Also define $\tilde{\boldsymbol{R}}_a$ as $\boldsymbol{R}_a$ in Eq. (13b), where r_i and r_{si} are replaced by $\tilde{r}_i$ and $\tilde{r}_{si}$, respectively. Finally, define $\tilde{G}$ as G in Eq. (26), where r_i is replaced by $\tilde{r}_i$. Using these definitions, and after dropping the last two terms (as was done in the development of the IDS), Eqs. (38) can now be written as

$$\tilde{H}\Delta\boldsymbol{X} = \tilde{\boldsymbol{R}}_a + \Delta\chi\boldsymbol{R}_b + \tilde{G}\boldsymbol{v} \tag{39}$$

As mentioned before, in the direct solution the pseudoinverse of Eq. (14) is used for solving Eqs. (10), which should actually be Eqs. (23). This implies the omission of the last three terms in Eqs. (23), which is unjustified. In contrast, neglecting the last two terms of Eqs. (38) is truly justified because of their smallness. As in the development of the IDS, and similar to Eq. (27), we can change Eq. (39) to read

$$\tilde{G}^{-1}(\tilde{\boldsymbol{R}}_a + \Delta\chi\boldsymbol{R}_b) = \tilde{G}^{-1}\tilde{H}\Delta\boldsymbol{X} - \boldsymbol{v} \tag{40}$$

from which we can compute the least-squares estimate of $\Delta\boldsymbol{X}$ as

$$\Delta\hat{\boldsymbol{X}} = [\tilde{H}^T\tilde{G}^{-2}\tilde{H}]^{-1}\tilde{H}^T\tilde{G}^{-2}(\tilde{\boldsymbol{R}}_a + \Delta\chi\boldsymbol{R}_b) \tag{41}$$

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be now

$$\tilde{\boldsymbol{p}} = [\tilde{H}^T\tilde{G}^{-2}\tilde{H}]^{-1}\tilde{H}^T\tilde{G}^{-2}\tilde{\boldsymbol{R}}_a \tag{42a}$$

$$\tilde{\boldsymbol{q}} = [\tilde{H}^T\tilde{G}^{-2}\tilde{H}]^{-1}\tilde{H}^T\tilde{G}^{-2}\boldsymbol{R}_b \tag{42b}$$

then Eq. (41) can be written as

$$\Delta\tilde{\boldsymbol{X}} = \tilde{\boldsymbol{p}} + \Delta\chi\tilde{\boldsymbol{q}} \tag{43}$$

From this point on, the computation proceeds along the algorithm of the direct solution; namely, $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{q}}$ are used in the computation of the coefficients of the quadratic equation in the direct solution [see Eq. (19)]. The two solutions of the quadratic equation are then used in Eq. (41) to compute two vectors, namely, $\Delta\hat{\boldsymbol{X}}_1$ and $\Delta\hat{\boldsymbol{X}}_2$, which are substituted into Eq. (34). One of the solutions almost satisfies the equation, whereas the other solution is way off. The former is then selected and added to $\hat{\boldsymbol{X}}$, our initial guess of $\boldsymbol{X}$. The computation is then repeated using the latest estimate of $\boldsymbol{X}$ as a starting point until convergence. The final estimate of $\boldsymbol{X}$ at this time will serve as the initial estimate at the next time. (As will be shown later, due to the fast convergence rate of this algorithm, there is no need to project forward the position components of the present estimate of $\boldsymbol{X}$ using velocity information.)

We can now formulate this process as follows.

1) Determine $\hat{\boldsymbol{X}}_0$, the initial estimate of $\boldsymbol{X}$.
2) Compute $\tilde{H}$, $\tilde{G}$, $\tilde{\boldsymbol{R}}_a$, and $\boldsymbol{R}_b$.
3) Compute $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{q}}$.

Table 1 Comparison of the computational burden

Algorithm	$\sqrt{\ }$	$\times/\div$	$\pm$	Remark
ILS	n[a]	$4n^2 + 27n + 40$	$3n^2 + 26n + 30$	——
Direct	1	$4n^2 + 31n + 70$	$3n^2 + 33n + 43$	Single iteration
IDS	1	$4n^2 + 41n + 70$	$3n^2 + 33n + 43$	——
IDSBIS	1	$4n^2 + 41n + 70$	$3n^2 + 33n + 51$	——

[a]Number of satellites.

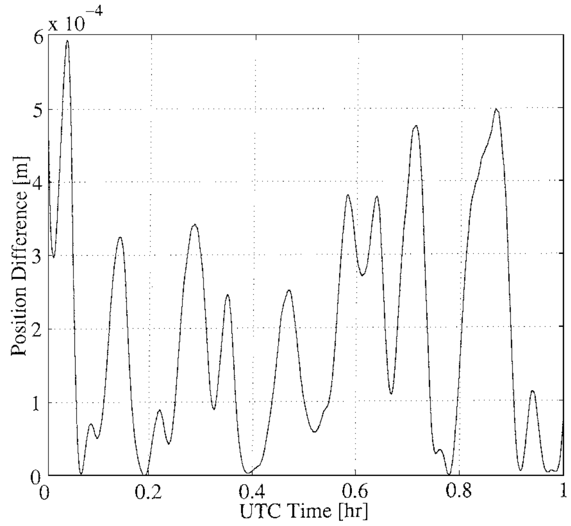

Fig. 3 Position difference between the ILS and the IDSBIS solutions: June 12, 1996.

4) Compute the coefficients of the quadratic equation [Eq. (19)].
5) Solve the quadratic equation for $\Delta\chi_1$ and $\Delta\chi_2$.
6) Use $\Delta\chi_1$ and $\Delta\chi_2$ to compute $\Delta\hat{\boldsymbol{X}}_1$ and $\Delta\hat{\boldsymbol{X}}_2$ using Eq. (41).
7) Select the $\Delta\hat{\boldsymbol{X}}_i$ $(i = 1, 2)$ that best satisfies Eq. (34) to be $\Delta\hat{\boldsymbol{X}}$.
8) Add $\Delta\hat{\boldsymbol{X}}$ to the last $\hat{\boldsymbol{X}}$.
9) If the current $\Delta\hat{\boldsymbol{X}}$ is closer than a predetermined ε to the preceding $\Delta\hat{\boldsymbol{X}}$, stop the iteration and move to the next step; otherwise return to step 2.
10) Proceed to the next time point and go back to step 2 using the latest $\hat{\boldsymbol{X}}$ (of the preceding time point) as the initial guess.

Figure 3 presents a comparison between the ILS solution and the IDSBIS solution when both were applied to the same data used earlier to examine the direct solution and the IDS. The largest difference between these two solutions is less than 0.6 mm, which means that for all practical purposes the accuracy of the ILS and the IDSBIS is the same. However, when the initial estimate of $\boldsymbol{X}$ was zero (the vehicle is at the center of Earth, and it is erroneously assumed that there is no clock bias), the ILS algorithm converged in five iterations, whereas the IDSBIS converged in only two iterations.

V. Discussion

Examination of the direct solution for a user's position when using actual GPS measurements revealed that the solution was worse than the popular ILS algorithm. The average position difference between the two reached 4 m. Consequently, we introduced an IDS, which is similar to the direct solution except that a certain pseudoinverse matrix is computed differently. Although this solution reduces the position deviation from the ILS solution by an order of magnitude, the IDS is still not optimal. Therefore, we proposed an iterative solution that is based on the IDS. The position accuracy of this algorithm (IDSBIS) is comparable to that of the ILS solution; however, the IDSBIS converges faster than the ILS solution. This is because the ILS algorithm ignores certain second-order error terms; therefore, initially, when the position error is large, it takes quite a few iterations for the ILS solution to reduce the position error to the range where dropping those error terms is justified. In contrast, the IDSBIS solution includes these second-order terms; thus, the first iteration already yields a close solution. The position difference between the IDSBIS solution and that of the classical ILS solution, using real data, was no larger than 6×10^{-4} m.

It is beneficial to compare the computational burden involved in each iteration of each approach. A comparison is presented in Table 1.

VI. Conclusion

Although the accuracy of the IDSBIS is comparable to that of the classical ILS solution, the IDSBIS is superior to the ILS solution in that it is a faster convergent algorithm. In a test that was carried out with real data, the ILS solution converged after five iterations, whereas the IDSBIS converged after only two. This difference is important in space navigation, especially when the GPS signal is reacquired after signal dropout, a situation where the error in the initial position estimate may be very large.

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